Conjoint measurement

or

How to measure preferences as you measure length

and

What to do if you cannot

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2022

Basic problem

Objects evaluated on several attributes

- helping a DM choose between these objects
- helping a DM structure his/her preferences (he in the sequel)

Classical decision analysis

- clear decision context
- careful analysis of objectives
- careful analysis of attributes
- careful selection of objects to be compared
- DM is available and proactive

Basic problem

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Context: decision analysis

- clear decision context
- careful analysis of objectives
- careful analysis of attributes
- careful selection of objects to be compared
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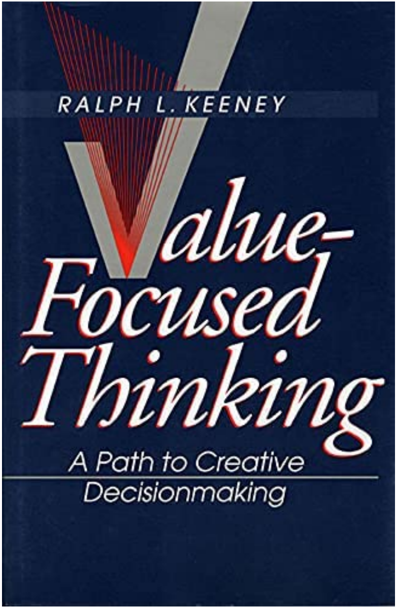
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Value-
Focused
Thinking
A Path to Creative
Decisionmaking

Decision with multiple attributes (MADM)

n attributes: $N = \{1, 2, \ldots, n\}$
x_i: evaluation of object x on attribute i
$v_i(x_i)$: numerical recoding using value function v_i

Basic model: Additive value function model

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} v_i(x_i) \geq \sum_{i=1}^{n} v_i(y_i)$$

- underlies many existing MADM techniques
- theory: additive conjoint measurement
- Economics (Debreu, 1960), Psychology (Luce & Tukey, 1964)

Outline

Outline

Aside: measurement of physical quantities

A collection of rigid straight rods

- problem: measuring the length of these rods

Three main steps

- comparing objects: experiment
- creating new objects: concatenation
- creating standard sequences

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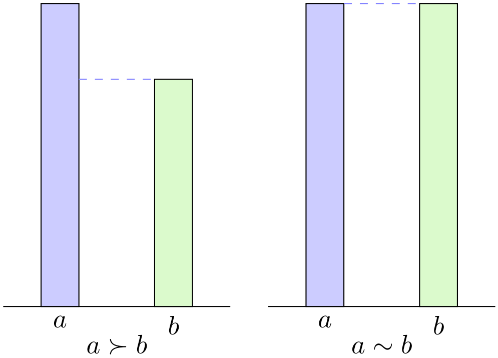
Step 1: comparing objects

- experiment to conclude which rod has "more length"
- place rods standing side by side on the same plane



Comparing objects

Results

- $a \succ b$: extremity of rod a is higher than extremity of rod b
- $a \sim b$: extremity of rod a is as high as extremity of rod b

Summary of experiments

- $\succsim \; = \; \succ \cup \sim$ is a weak order
  - complete ($a \succsim b$ or $b \succsim a$)
  - transitive ($a \succsim b$ and $b \succsim c \Rightarrow a \succsim c$)
    - both $\succ$ and $\sim$ are transitive
    - they combine in a nice way: $a \succ b$ and $b \sim c \Rightarrow a \succ c$

Comparing objects

Consequences

$\succsim = \succ \cup \sim$ is a weak order

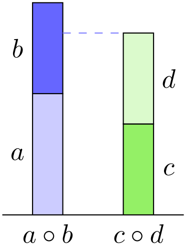
- associate a real number $\Phi(a)$ to each object a (under mild conditions)
- the comparison of numbers faithfully reflects the results of experiments

$$a \succ b \Leftrightarrow \Phi(a) > \Phi(b)$$
$$a \sim b \Leftrightarrow \Phi(a) = \Phi(b)$$

- the function Φ defines an ordinal scale
- defined up to an increasing transformation

Step 2: creating and comparing new objects

- use the available objects to create new ones
- concatenate objects by placing two or more rods "in a row"



$$a \circ b \succ c \circ d$$

Concatenation

Constraints induced by concatenation

- we want to be able to deduce $\Phi(a \circ b)$ from $\Phi(a)$ and $\Phi(b)$
- simplest requirement

$$\Phi(a \circ b) = \Phi(a) + \Phi(b)$$

- monotonicity constraints

$$\left.\begin{array}{l} a \succ b \\ c \sim d \end{array}\right\} \Rightarrow a \circ c \succ b \circ d$$

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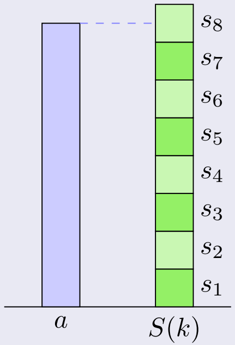
Step 3: creating and using standard sequences

- choose a standard rod: our mètre étalon
- be able to build perfect copies of the standard
- concatenate the standard rod with its perfect copies



$$S(8) \succ a \succ S(7)$$

$$\Phi(s) = 1 \Rightarrow 7 < \Phi(a) < 8$$

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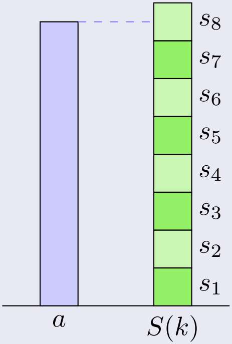

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Convergence

First method

- choose a smaller standard rod
- repeat the process

Second method

- prepare a perfect copy of the object
- concatenate the object with its perfect copy
- compare the “doubled” object to the original standard sequence
- repeat the process

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Scale of length

- ratio scale: natural zero and change of unit only ($\Phi' = \alpha\Phi, \alpha > 0$)

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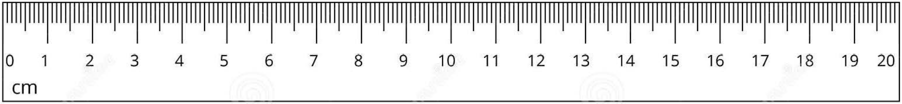

Summary

Extensive measurement (Archimedean Ordered Semi-groups)

- Krantz, Luce, Suppes & Tversky, 1971, chap. 3

Four technical ingredients

1. well-behaved relations $\succ$ and $\sim$
2. concatenation operation $\circ$
3. consistency requirements linking $\succ$, $\sim$ and $\circ$
4. preparing perfect copies of some objects to build standard sequences

Neglected problems

- many!
- idealization of the measurement process

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Question

Can this be applied outside Physics?

- no concatenation operation
    - intelligence!
    - pain!

Outline

1. Preliminaries: length and even swaps
   - An aside: measurement in Physics
   - An simple example: even swaps
2. Additive value functions: outline of theory
3. Additive value functions: implementation
4. A glimpse at possible extensions

"The best book I know on how to make a decision."
—Roger Fisher, coauthor of the bestseller *Getting to Yes*

Smart Choices

A PRACTICAL GUIDE TO MAKING BETTER LIFE DECISIONS

JOHN S. HAMMOND | RALPH L. KEENEY | HOWARD RAIFFA

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Even swaps: Hammond, Keeney & Raiffa, 1999

Independent consultant: choice of an office to rent

- five locations have been identified
- five attributes are being considered
  - *Commute* time (minutes)
  - *Clients*: percentage of clients living close to the office
  - *Services*: ad hoc scale
    - A (all facilities), B (water & heating), C (no facility)
  - *Size*: square feet ($\simeq 0.1$ m^2)
  - *Cost*: \$ per month

"Data"

	a	b	c	d	e
Commute	45	25	20	25	30
Clients	50	80	70	85	75
Services	A	B	C	A	C
Size	800	700	500	950	700
Cost	1850	1700	1500	1900	1750

Hypothesis

- ceteris paribus reasoning seems possible
  Commute: decreasing *Clients*: increasing
  Services: increasing *Size*: increasing
  Cost: decreasing
- dominance has meaning

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- d is "close" to dominating a
- divide and conquer: dropping alternatives
  - drop a and e

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- no more dominance
- assessing tradeoffs
- all alternatives except c have a common evaluation on *Commute*
- modify c in order to bring it to this level
  - starting with c, what is the gain on *Clients* that would exactly compensate a loss of 5 min on *Commute*?
  - difficult but crucial question
  - clear context, structured objectives, bracketing

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ARTICLES

COMMON MISTAKES IN MAKING VALUE TRADE-OFFS

RALPH L. KEENEY

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(Received September 2000; revision received April 2001; accepted November 2001)

	c	c'
Commute	20	**25**
Clients	70	$\mathbf{70} + \delta$
Services	C	C
Size	500	500
Cost	1500	1500

find δ such that $c' \sim c$

Answer

- for $\delta = 8$, I am indifferent between c and c'
- replace c with c'

	c	c'
Commute	20	**25**
Clients	70	$\mathbf{70} + \delta$
Services	C	C
Size	500	500
Cost	1500	1500

find δ such that $c' \sim c$

Answer

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	b	c'	d
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- all alternatives have a common evaluation on *Commute*
- divide and conquer: dropping attributes
  - drop attribute *Commute*

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Two phases: divide and conquer

- check for dominance: drop alternatives
- assessing tradeoff: drop attributes
  - last step is always a dominance step leading to the recommended choice
  - very simple and quite transparent process

Summary: Even Swaps

Remarks

- very simple process: the polar technique in MADM
- process entirely governed by $\succ$ and $\sim$ (the primitives)
- notice that importance is not even mentioned
  - it is there but in a more complex form than just "weights"
- why be interested in something more complex?

Problems (by increasing order of importance)

- set of alternative is small
  - many questions otherwise
- output is not a preference model
  - if new alternatives appear, the process should be restarted
- what are the underlying hypotheses?

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Primitives as the super-ego of the decision theorist

Constraint

- you can only phrase questions to the DM in terms of the primitives

Remark

- you do not suppose that you know the primitives...
- ... but you suppose that questions on the primitives can be meaningfully answered (if asked properly)

Choice of primitives

$\succsim$: consensus primitives in decision theory
counterexamples: AHP, MACBETH, Choquet

Monsieur Jourdain doing conjoint measurement 1/2

Similarity with extensive measurement

- $\succ$: preference, $\sim$: indifference
- we have implicitly supposed that they combine nicely $(c' \sim c)$
- standard sequences are hidden but they are present

Monsieur Jourdain doing conjoint measurement 2/2

	c	c'	f	f'
Commute	20	25	20	25
Clients	70	**78**	78	82
Services	C	C	C	C
Size	500	500	500	500
Cost	1500	1500	1500	1500

$c \sim c' \Rightarrow [70, 78]$ has the same length as the reference $[20, 25]$

$f \sim f' \Rightarrow [78, 82]$ has the same length as the reference $[20, 25]$

$\Rightarrow [70, 78]$ has the same length as $[78, 82]$ on *Client*

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Monsieur Jourdain doing conjoint measurement 2/2

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$$c \sim c' \Rightarrow [70, 78] \text{ has the same length as the reference } [20, 25]$$
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$$\Rightarrow [70, 78] \text{ has the same length as } [78, 82] \text{ on } \textit{Client}$$

Outline

Setting

- $N = \{1, 2, \ldots, n\}$ set of attributes
- X_i: set of possible levels on the ith attribute
- $X = \prod_{i=1}^{n} X_i$: set of all conceivable alternatives
  - X include the alternatives under study… and many others

- $J \subseteq N$: subset of attributes
- $X_J = \prod_{j \in J} X_j$, $X_{-J} = \prod_{j \notin J} X_j$
- $(x_J, y_{-J}) \in X$
- $(x_i, y_{-i}) \in X$

- $\succsim$: binary relation on X: “at least as good as”
  - the primitives
- $x \succ y \Leftrightarrow x \succsim y$ and $Not[y \succsim x]$
- $x \sim y \Leftrightarrow x \succsim y$ and $y \succsim x$

Independence

Independence

- J is independent for $\succsim$ if

$$[(x_J, w_{-J}) \succsim (y_J, w_{-J}), \text{ for some } w_{-J} \in X_{-J}] \Rightarrow$$
$$[(x_J, z_{-J}) \succsim (y_J, z_{-J}), \text{ for all } z_{-J} \in X_{-J}]$$

- common levels on attributes other than J do not affect preference
- tradeoffs in J do not depend on the common levels on $-J$
- notation for independent subset J: $x_J \succsim_J y_J$

Independence

Definition

- for all $i \in N$, $\{i\}$ is independent, $\succsim$ is weakly independent
- for all $J \subseteq N$, J is independent, $\succsim$ is independent

- independence $\Rightarrow$ weak independence
- identical for $n = 2$

Proposition (Folk lemma for dominance)

Let $\succsim$ be a weakly independent weak order on $X = \prod_{i=1}^{n} X_i$. Then:

- $\succsim_i$ is a weak order on X_i
- $[x_i \succsim_i y_i$, for all $i \in N] \Rightarrow x \succsim y$
- $[x_i \succsim_i y_i$, for all $i \in N$ and $x_j \succ_j y_j$ for some $j \in N] \Rightarrow x \succ y$

for all $x, y \in X$

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for all $x, y \in X$

Independence in practice

Independence

- it is easy to imagine examples in which independence is violated
    - Main course and Wine (too often used) example

$$(\text{Fish}, \text{WW}) \succ (\text{Meat}, \text{WW})$$
$$(\text{Meat}, \text{RW}) \succ (\text{Fish}, \text{RW})$$

- it is nearly hopeless to try to work if weak independence is not satisfied

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$$(\text{Meat}, \text{RW}) \succ (\text{Fish}, \text{RW})$$

- it is nearly hopeless to try to work if weak independence is not satisfied
- some (e.g., R. L. Keeney) think that the same is true for independence
- in all cases if independence is violated, things get complicated

Outline of theory: 2 attributes

Question

- primitives: $\succsim$ on $X = X_1 \times X_2$
- what must be supposed to guarantee that I can represent $\succsim$ in the additive value function model

$$v_1 : X_1 \to \mathbb{R}$$
$$v_2 : X_2 \to \mathbb{R}$$
$$(x_1, x_2) \succsim (y_1, y_2) \Leftrightarrow v_1(x_1) + v_2(x_2) \geq v_1(y_1) + v_2(y_2)$$

- $\succsim$ must be an independent weak order
- is this sufficient?

Method

- try building standard sequences and see if it works!

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Outline of theory: 2 attributes

Question

- primitives: $\succsim$ on $X = X_1 \times X_2$
- what must be supposed to guarantee that I can represent $\succsim$ in the additive value function model

$$v_1 : X_1 \to \mathbb{R}$$
$$v_2 : X_2 \to \mathbb{R}$$
$$(x_1, x_2) \succsim (y_1, y_2) \Leftrightarrow v_1(x_1) + v_2(x_2) \geq v_1(y_1) + v_2(y_2)$$

- $\succsim$ must be an independent weak order
- is this sufficient?

Method

- try building standard sequences and see if it works!

Uniqueness

Important observation

Suppose that there are v_1 and v_2 such that

$$(x_1, x_2) \succsim (y_1, y_2) \Leftrightarrow v_1(x_1) + v_2(x_2) \geq v_1(y_1) + v_2(y_2)$$

Take $\alpha, \beta_1, \beta_2 \in \mathbb{R}$ with $\alpha > 0$

$$w_1 = \alpha v_1 + \beta_1 \quad w_2 = \alpha v_2 + \beta_2$$

is also a valid representation

$$v_1(x_1) + v_2(x_2) \geq v_1(y_1) + v_2(y_2) \Leftrightarrow$$

$$\big(\alpha v_1(x_1) + \beta_1\big) + \big(\alpha v_2(x_2) + \beta_2\big) \geq \big(\alpha v_1(y_1) + \beta_1\big) + \big(\alpha v_2(y_2) + \beta_2\big)$$

Consequences

- fixing $v_1(x_1^0) = v_2(x_2^0) = 0$ is harmless
- fixing $v_1(x_1^1) = 1$ is harmless if $x_1^1 \succ_1 x_1^0$
- preference interval $[x_1^0, x_1^1]$ will be our standard rod, our mètre étalon
- $v_1(x_1^1) - v_1(x_1^0) = 1$

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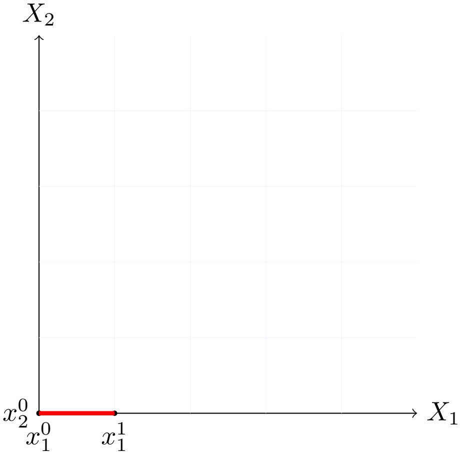
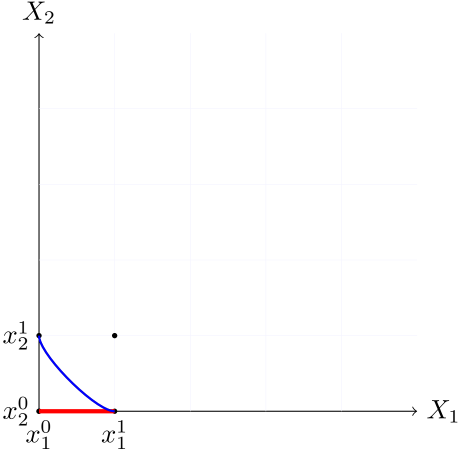
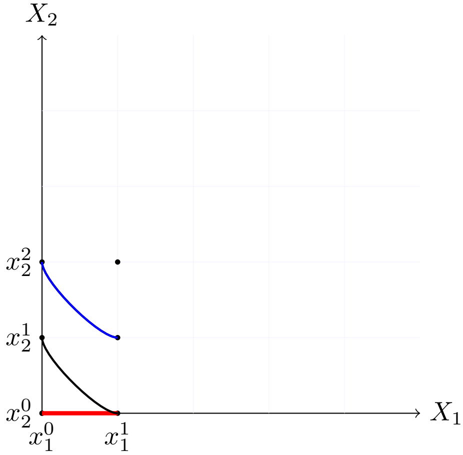


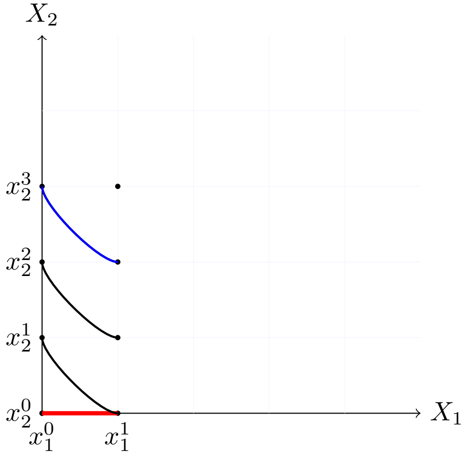
X_2
X_1
x_2^0
x_1^0
x_1^1



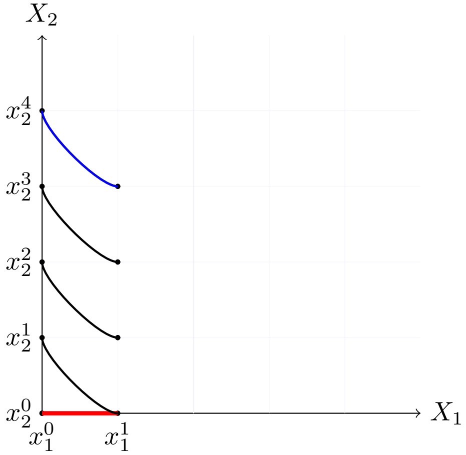
$$(x_1^1, x_2^0) \sim (x_1^0, x_2^1)$$



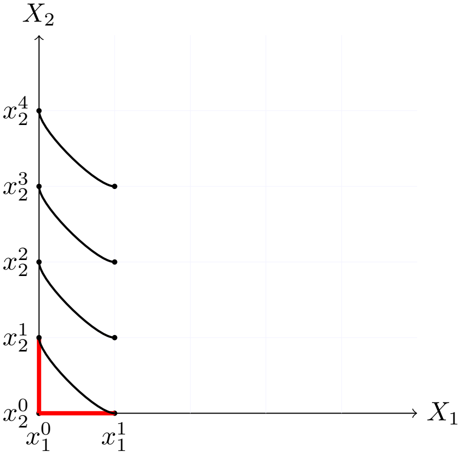
$$(x_1^1, x_2^1) \sim (x_1^0, x_2^2)$$



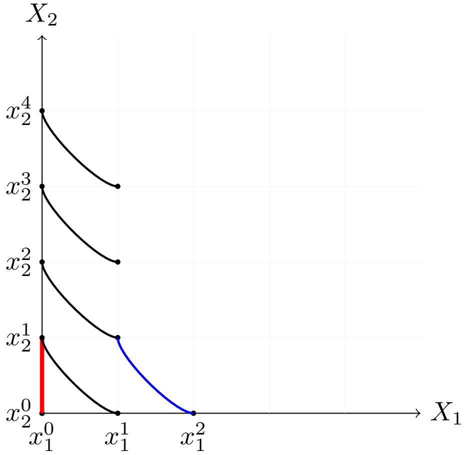
$$(x_1^1, x_2^2) \sim (x_1^0, x_2^3)$$



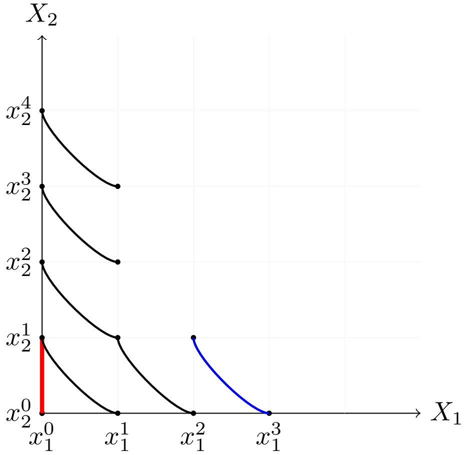
$$(x_1^1, x_2^3) \sim (x_1^0, x_2^4)$$



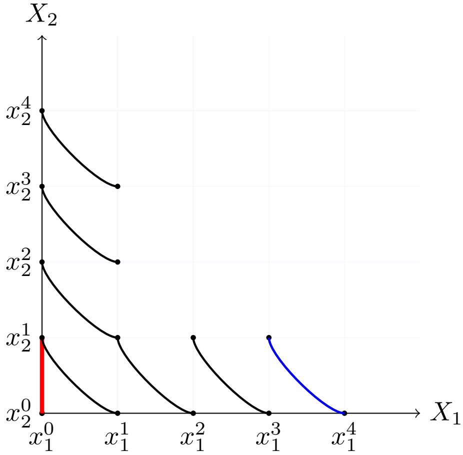
$$(x_1^1, x_2^0) \sim (x_1^0, x_2^1)$$



$$(x_1^2, x_2^0) \sim (x_1^1, x_2^1)$$



$$(x_1^3, x_2^0) \sim (x_1^2, x_2^1)$$



$$(x_1^4, x_2^0) \sim (x_1^3, x_2^1)$$

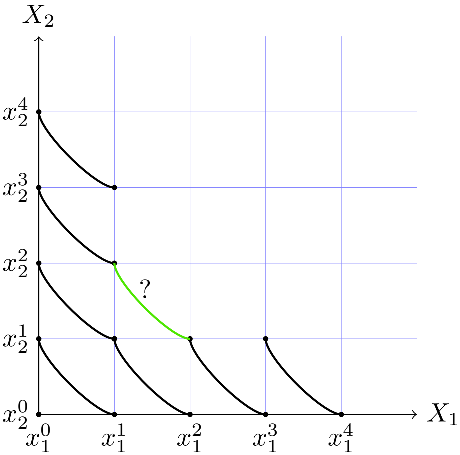
Standard sequence

Archimedean

- implicit hypothesis for length
  - the standard sequence can reach the length of any object

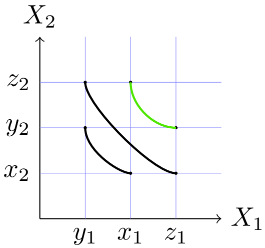
$$\forall x, y \in \mathbb{R}, x, y > 0, \exists n \in \mathbb{N} : ny > x$$

- a similar hypothesis has to hold here
- rough interpretation
  - there are not "infinitely" liked or disliked consequences


X_2
x_2^4
x_2^3
x_2^2
x_2^1
x_2^0
?
X_1
x_1^0
x_1^1
x_1^2
x_1^3
x_1^4

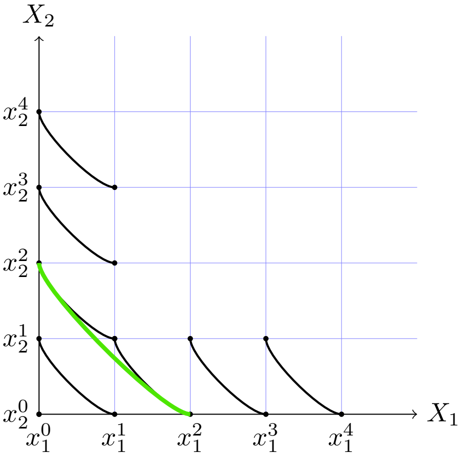
Thomsen condition

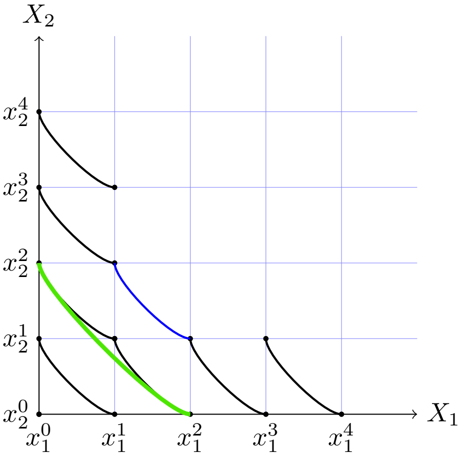
$$\left.\begin{array}{c}(x_1, x_2) \sim (y_1, y_2) \\ \text{and} \\ (y_1, z_2) \sim (z_1, x_2)\end{array}\right. \Rightarrow (x_1, z_2) \sim (z_1, y_2)$$



Consequence

- there is an additive value function on the grid


X_2
x_2^4
x_2^3
x_2^2
x_2^1
x_2^0
X_1
x_1^0
x_1^1
x_1^2
x_1^3
x_1^4


X_2
x_2^4
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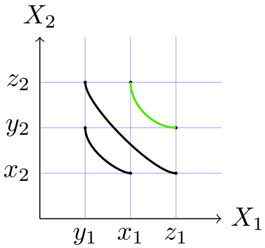

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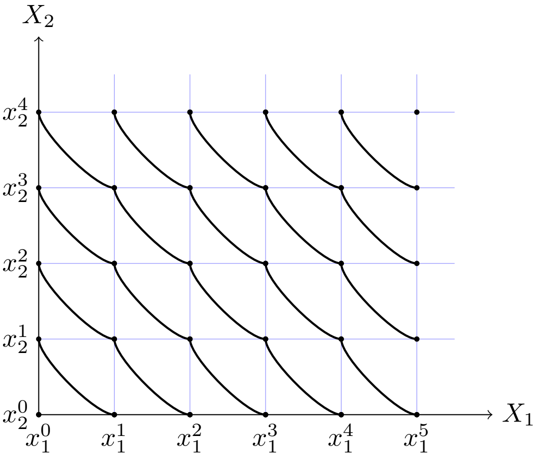

X_2
x_2^4
x_2^3
x_2^2
x_2^1
x_2^0
X_1
x_1^0
x_1^1
x_1^2
x_1^3
x_1^4
x_1^5

Summary

- we have defined a “grid”
- there is an additive value function on the grid

- iterate the whole process with a “denser grid”

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Additional hypotheses

- Archimedean (no infinitely liked or disliked consequence)
- restricted solvability (richness of X_1 and X_2)
- essentiality (both attributes affect preference)

Basic result

Theorem (2 attributes)

If

- restricted solvability holds
- each attribute is essential

then
the additive value function model holds
if and only if
$\succsim$ is an independent weak order satisfying the Thomsen and the Archimedean conditions

The representation is unique up to scale and location

Interval scales with a common unit

If $\langle v_1, v_2 \rangle$ and $\langle w_1, w_2 \rangle$ both represent $\succsim$ in the additive value function model in the context of the above result, then there are $\alpha > 0$ and $\beta_1, \beta_2 \in \mathbb{R}$, such that

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General case

Good news

- entirely similar...
- with a very nice surprise: Thomsen can be forgotten
  - if $n = 2$, independence is identical with weak independence
  - if $n \geq 3$, independence is much stronger than weak independence

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Basic result

Theorem (more than 2 attributes)

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- restricted solvability holds
- at least three attributes are essential

then
the additive value function model holds
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$\succsim$ is an independent weak order satisfying the Archimedean condition

The representation is unique up to scale and location

Independence and even swaps

Even swaps technique

- assessing tradeoffs. . .
- after having suppressed attributes

Implicit hypothesis

- what happens on these attributes do not influence tradeoffs
- this is another way to formulate independence

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Foundations of even swaps technique

Hypotheses

- weak order
- independence
- unrestricted solvability on one attribute

Close but not equivalent to the additive value function model!

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Outline

1. Preliminaries: length and even swaps
2. Additive value functions: outline of theory
3. Additive value functions: implementation
    - Full assessment
    - Misconceptions
    - Partial assessment
4. A glimpse at possible extensions

Assessing additive value functions

Full assessment

- in a "rich" setting (i.e., with solvability)
- check independence
- build standard sequences
  - use even swaps if you do not need an explicit model or if solvability only holds on one attribute

Possible problems

- many questions
- discrete attributes (no solvability)
- propagation of "errors"

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Dirty tricks

Do not stick to the primitives of the model

- bisection
- direct rating
- intensity of preference
- questions about importance

No dirty tricks here!

- they will send you directly to decision theory hell!
- they may work but there is no clear foundation

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DECISION ANALYSIS AND BEHAVIORAL RESEARCH

Detlof von Winterfeldt

Ward Edwards

Misconceptions I: weights and normalization

Critical mistake

- use the “weighted sum” as a degraded version of the additive value function model
- analogy: French baccalauréat (weighted sum of grades from 0 to 20 with threshold at 10)
- MADM = normalize and weigh

This is plain nonsense!

Normalize and weigh

Equal weights

	f_1	f_2	f_1'	f_2'	S	f_1''	f_2''	T
A	2 000	500	100	100	100.0	100	100	100.0
B	1 120	175	56	35	45.5	70	35	52.5
C	400	370	20	74	47.0	25	74	49.5
D	1 600	45	80	9	44.5	100	9	54.5
E	880	240	44	48	46.0	55	48	51.5
F	160	435	8	87	47.5	10	87	48.5
G	1 360	110	68	22	45.0	85	22	53.5
H	640	305	32	61	46.5	40	61	50.5

$$A \succ F \succ C \succ H \succ E \succ B \succ G \succ D$$

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Normalize and weigh

Changing the performance of A . . .

. . . reverses the ranking of all other alternatives!

Weights in a weighted sum

- scaling constants
- they do not indicate intrinsic importance
- they are linked to the width of the scale
  - $(2\,000, 0) \sim (0, 500)$ or $(1\,600, 0) \sim (0, 500)$?
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Misconceptions II: independence entails additivity

Critical mistake

- independence is necessary but not sufficient!
- this is only so in a well-behaved continuous setting

Example: finite structure

$X = X_1 \times X_2$
$X_1 = \{a, b, c\}$
$X_2 = \{d, e, f\}$

$$ad \succ bd \succ ae \succ af \succ be \succ cd \succ ce \succ bf \succ cf$$

- independence is satisfied, e.g., we have:

$$ad \succ bd \succ cd \text{ and } ae \succ be \succ ce \text{ and } af \succ bf \succ cf$$

$$ad \succ ae \succ af \text{ and } bd \succ be \succ bf \text{ and } cd \succ ce \succ cf$$

- Thomsen is trivially satisfied: no indifference
- Archimedean is trivially satisfied in a finite structure
  - "every strictly bounded standard sequence is finite"

Solvability is violated

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$$bd \succ ae \Rightarrow v_1(b) + v_2(d) > v_1(a) + v_2(e)$$

$$(1):\ v_1(a) + v_2(f) > v_1(c) + v_2(d)$$

$$(2\text{–}3):\ v_1(c) + v_2(d) > v_1(a) + v_2(f)$$

Conclusion

No additive value function can represent this independent weak order satisfying Thomsen and Archimedean

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Misconceptions III: the finite case is simpler

The finite case is way more complex

- our central tool (standard sequences) is lost
  - measuring length in a discrete space
- nice uniqueness results cannot be expected

$$x \succ y \Rightarrow \sum_{i=1}^{n} v_i(x_i) > \sum_{i=1}^{n} v_i(y_i)$$

$$x \sim y \Rightarrow \sum_{i=1}^{n} v_i(x_i) = \sum_{i=1}^{n} v_i(y_i)$$

- apply criteria for solvability of a system of linear equations and inequalities
  - one form of theorem of the alternative
- one unknown ($v_i(x_i)$) per element of each $x_i \in X_i$
- denumerable scheme of conditions that cannot be truncated
- in practice, use LP: UTA

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Assessing a set of additive utility functions for multicriteria decision-making, the UTA method

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Received August 1981
Revised November 1981

Using LP as a tool for MADM (and not MODM)

Outline

1. Preliminaries: length and even swaps
2. Additive value functions: outline of theory
3. Additive value functions: implementation
4. A glimpse at possible extensions
   - Models with interactions
   - Nontransitive models
   - New primitives

Summary

Additive value function model

- conceptually simple
- technically close to extensive measurement

General techniques

- standard sequences
- even swaps, UTA

Additive value function model

- requires independence
- requires a finely grained analysis of preferences

Three main types of extensions

1. abandon independence: models "with interactions"
2. abandon finely grained analysis: "ordinal" models
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- even swaps, UTA

Additive value function model

- requires independence
- requires a finely grained analysis of preferences

Three main types of extensions

1. abandon independence: models “with interactions”
2. abandon finely grained analysis: “ordinal” models
3. new primitives

Misconceptions IV

Conjoint measurement is limited to additive conjoint measurement

Conjoint measurement

- can be extended in many different directions
- can provide solid decision-theoretic foundations to many models
- provides a general framework for comparing MADM techniques

Outline

Decomposable model

$$x \succsim y \Leftrightarrow F[v_1(x_1), \ldots, v_n(x_n)] \geq F[v_1(y_1), \ldots, v_n(y_n)]$$

F increasing in all arguments

Decision-theoretic foundations

- weakly independent weak order (KLST, 1971)
- case of F non-decreasing (B., Pirlot, 2004)
- strong links with models using reference points (B., Marchant, 2013)

Problems

- all possible types of interactions are admitted
- assessment is a very challenging task (DRSA)
- each v_i is an ordinal scale

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Models with interactions

Models in between additive and decomposable

- Polynomial models (KLST, 1971, ch. 7)
- GAI and extensions
- CP-nets and extensions
- Choquet integral (?)

Difficulties

- each line is a whole family of models
- difficult to propose general decision-theoretic foundations
  - diagnostic questions to specify a precise model
  - find adequate general restrictions (2-additivity for Choquet integral)

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go faster

GAI: Example

Choice of a meal: 3 attributes

$X_1 = \{\text{Steak}, \text{Fish}\}$
$X_2 = \{\text{Red}, \text{White}\}$
$X_3 = \{\text{Cake}, \text{sherBet}\}$

Preferences

$$\begin{array}{cccc} x^1 = (S, R, C) & x^2 = (S, R, B) & x^3 = (S, W, C) & x^4 = (S, W, B) \\ x^5 = (F, R, C) & x^6 = (F, R, B) & x^7 = (F, W, C) & x^8 = (F, W, B) \end{array}$$

$$x^2 \succ x^1 \succ x^7 \succ x^8 \succ x^4 \succ x^3 \succ x^5 \succ x^6$$

- the important is to match main course and wine
- I fancy Steak today
- I prefer Cake to sherBet if Fish
- I prefer sherBet to Cake if Steak

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Independence

$$x^1 \succ x^5 \Rightarrow v_1(S) > v_1(F)$$
$$x^7 \succ x^3 \Rightarrow v_1(F) > v_1(S)$$

Grouping main course and wine?

$$x^7 \succ x^8 \Rightarrow v_3(C) > v_3(B)$$
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$$x \succsim y \Leftrightarrow v_{12}(x_1, x_2) + v_{13}(x_1, x_3) \geq v_{12}(y_1, y_2) + v_{13}(y_1, y_3)$$

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GAI (Gonzales & Perny)

- first axiomatic analysis
- if interdependencies are known
  - assessment techniques
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Outline

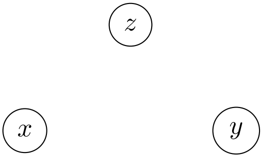
Nontransitive models

Ordinal models

- no finely grained analysis of preferences
- close to social choice
- aggregating preference relations (without anonymity)
- ELECTRE methods and related outranking approaches
- nontransitive preferences (Condorcet effect)

$$1 : x_1 \succ_1 y_1 \succ_1 z_1$$
$$2 : z_2 \succ_2 x_2 \succ_2 y_2$$
$$3 : y_3 \succ_3 z_3 \succ_3 x_3$$

$$x = (x_1, x_2, x_3)$$
$$y = (y_1, y_2, y_3)$$
$$z = (z_1, z_2, z_3)$$



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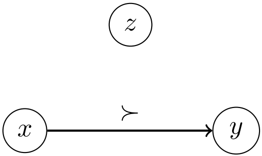

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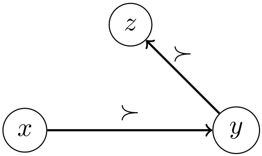

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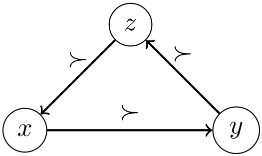

Nontransitive decomposable conjoint measurement

Ordinal models and conjoint measurement

- nontransitive decomposable conjoint measurement
- foundations of concordance / non-discordance relations à la ELECTRE

(B., Pirlot 2002, 2004a,b,c, 2006, 2008, 2015a,b)

$$x \succsim y \Leftrightarrow G(p_1(x_1, y_1), p_2(x_2, y_2), \ldots, p_n(x_n, y_n)) \geq 0$$

- G nondecreasing in all arguments
- p_i skew-symmetric: "preference differences"

- if p_i takes at most three distinct values: concordance relations
- if p_i takes at most five distinct values: concordance-discordance relations

Variants

- G odd
- $p_i(x_i, y_i) = \phi_i(v_i(x_i, v_i(y_i))$ with $\phi_i(\nearrow, \searrow)$

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Nontransitive models menagerie

$$x \succsim y \Leftrightarrow G(\langle p_i(x_i, y_i)\rangle) \geq 0$$
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$$x \succsim y \Leftrightarrow G(\langle \Phi_i(v_i(x_i) - v_i(y_i))\rangle) \geq 0$$
$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} p_i(x_i, y_i) \geq 0$$
$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} \Phi_i(v_i(x_i) - v_i(y_i)) \geq 0$$

p_i shew-symmetric
G increasing in all arguments, G odd, $\phi_i(\nearrow, \searrow)$
$\Phi_i(\nearrow)$, Φ_i odd

Outline

Sorting: new primitives

Primitives for sorting

- Twofold ordered partition of X: $\langle \mathcal{A}, \mathcal{U} \rangle$
- $x \in \mathcal{A}$ is "sAtisfactory"
- $x \in \mathcal{U}$ is "Unsatisfactory"

- can easily be generalized to more than two categories

Additive value function model for sorting

$$x \in \mathcal{A} \Leftrightarrow \sum_{i=1}^{n} u_i(x_i) > \lambda$$

- full decision-theoretic foundations for UTADIS-like models
  - rich case
  - finite case

(B. & Marchant, 2008, 2009)

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(B. & Marchant, 2008, 2009)

Decomposable model for sorting

Model

$$x \in \mathcal{A} \Leftrightarrow F(v_1(x_1), v_2(x_2), \ldots, v_n(x_n)) > \lambda$$

F increasing or nondecreasing
(Goldstein, 1991)

- complete decision-theoretic foundations: linearity
- assessment if a formidable task
- fully equivalent to ELECTRE TRI-nB (B., Marchant, Pirlot, 2022a,b)

Noncompensatory models for sorting

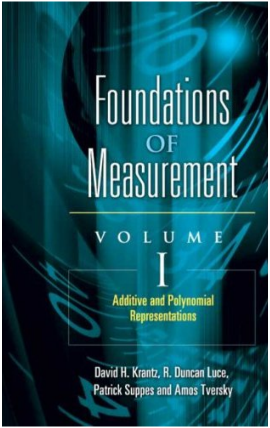
Model

$$x \in \mathcal{A} \Leftrightarrow F(v_1(x_1), v_2(x_2), \ldots, v_n(x_n)) > \lambda$$

F nondecreasing
v_i taking at most two values
F distinguishes winning from loosing coalitions

- decision-theoretic foundations for noncompensatory sorting models
- quite close to ELECTRE TRI-B
- related models: MR-Sort
- decision-theoretic foundations for the Sugeno integral

(B., Marchant, 2007a,b, B., Marchant, Pirlot, 2009)


Foundations
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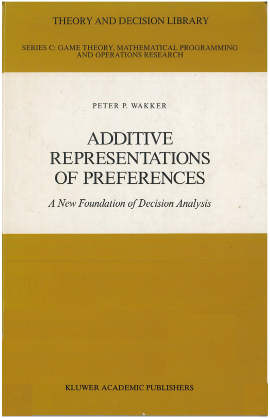
Measurement
VOLUME
I
Additive and Polynomial
Representations
David H. Krantz, R. Duncan Luce,
Patrick Suppes and Amos Tversky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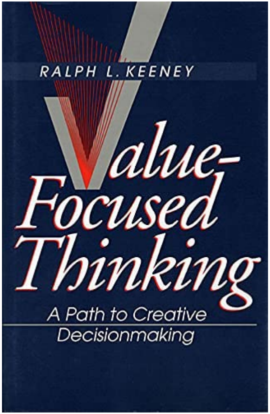
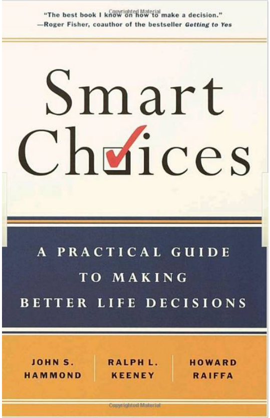

RALPH L. KEENEY
Value-Focused Thinking
A Path to Creative Decisionmaking


"The best book I know on how to make a decision."
—Roger Fisher, coauthor of the bestseller Getting to Yes
Smart Choices
A PRACTICAL GUIDE TO MAKING BETTER LIFE DECISIONS
JOHN S. HAMMOND
RALPH L. KEENEY
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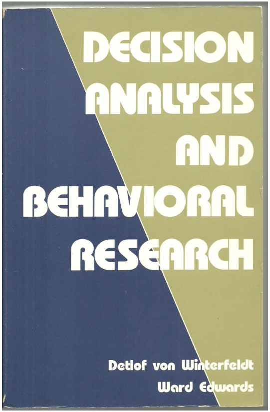
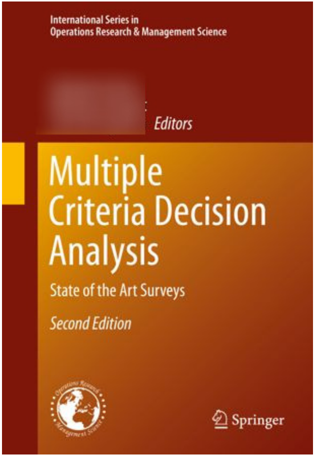

DECISION ANALYSIS AND BEHAVIORAL RESEARCH
Detlof von Winterfeldt
Ward Edwards



Chapter 4
Conjoint Measurement Tools for MCDM

A Brief Introduction

Denis Bouyssou and Marc Pirlot

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go to end and skip other references

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C'EST TOUT POUR AUJOURD'HUI.

C'EST TOUT POUR LE MOMENT